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LETTER TO THE EDITOR

On the spectrum of $S = \frac{1}{2}$ XXX Heisenberg chain with elliptic exchange

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Abstract. It is found that the Hamiltonian of the $S = \frac{1}{2}$ isotropic Heisenberg chain with N sites and elliptic non-nearest-neighbour exchange is diagonalized in each sector of the Hilbert space with magnetization $N/2 - M$, $1 < M \leqslant [N/2]$, by means of double quasiperiodic meromorphic solutions to the M-particle quantum Calogero–Moser problem on a line. The spectrum and highest-weight states are determined by the solutions of the systems of transcendental Bethe-ansatz-type equations which arise as restrictions on particle pseudomomenta.

In recent years, much attention has been paid to studies of 1D lattice systems, due to their relevance to principal notions of field theory and experimental investigations of effectively low-dimensional crystals. Even the simplest lattice systems, namely isotropic $S = \frac{1}{2}$ Heisenberg chains, have unveiled a rich structure and have provided non-trivial examples of many-body interactions. The corresponding mathematical problem consists of finding the proper analytic tool for the diagonalization of the model Hamiltonian

$$\mathcal{H}^{(s)} = \frac{J}{4} \sum_{1 \leqslant j \neq k \leqslant N} h(j-k)(\sigma_j \sigma_k - 1) \qquad h(j) = h(j+N) \tag{1}$$

where σ_j are Pauli matrices acting on the spin at the jth site.

At finite N, it has been successfully treated in the integrable cases of nearest-neighbour coupling solved by Bethe [1]

$$h(j) = \delta_{|j(\mathrm{mod}N)|,1} \tag{2}$$

and long-range trigonometric exchange proposed independently by Haldane and Shastry [2]

$$h(j) = \left(\frac{N}{\pi} \sin \frac{\pi j}{N}\right)^{-2}. \tag{3}$$

At present, a number of impressive results are known for both these models. In particular, they include the additivity of the spectrum under proper choice of 'rapidity' variables [1, 3], the description of the underlying symmetry [4, 5], the construction of thermodynamics in the limit $N \to \infty$ [6, 3], the connection with the continuum integrable many-body problems [7, 2], and closed-form expressions of correlations in the antiferromagnetic ground state. The rich collection of various generalizations and physical applications of the Bethe and Haldane–Shastry models can be found in recent review papers [8, 9].

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Several years ago, I introduced a more general one-parametric form of spin exchange which provides another example of the integrable lattice Hamiltonian (1) [10]. It was motivated by the similarity of the Lax representation of the Heisenberg equations of motion for the continuum and lattice models. In the former case, the most general translationally-invariant integrable Hamiltonian with elliptic pairwise particle interaction was found by Calogero [11] and Moser [12]:

$$H_{\rm CM} = \frac{1}{2}\left[-\sum_{\beta=1}^{L}\frac{\partial^2}{\partial x_\beta^2} + \lambda(\lambda+1)\sum_{\beta\neq\gamma}^{L}\wp(x_\beta - x_\gamma)\right]. \tag{4}$$

The existence of extra integrals of motion commuting with (4) was demonstrated in [13]. Recently, the eigenvalue problem for the elliptic Calogero–Moser operator received much attention due to its relation to the representations of double affine algebras and the solutions of Knizhnik–Zamolodchikov–Bernard equations [14, 15].

The lattice analogue of (4) is given by (1) with

$$h(j) = \left(\frac{\omega}{\pi}\sin\frac{\pi}{\omega}\right)^2\left[\wp_N(j) + \frac{2}{\omega}\zeta_N\left(\frac{\omega}{2}\right)\right] \tag{5}$$

where $\wp_N(x)$, $\zeta_N(x)$ are the Weierstrass functions defined on the torus $T_N = \mathbf{C}/\mathbf{Z}N + \mathbf{Z}\omega$, $\omega = \mathrm{i}\kappa$, $\kappa \in \mathbb{R}_+$. Remarkably, it turned out that the exchange (5) comprises both (2) and (3) [10]: in fact, the factor in (5) is chosen so as to reproduce the nearest-neighbour coupling under periodic boundary conditions (2) in the limit $\kappa \to 0$ and the long-range exchange (3) in the limit $\kappa \to \infty$.

However, up to now much less has been known about the lattice model with the exchange (5) compared with its limiting forms, due to the mathematical complexities caused by the presence of the Weierstrass functions. The family of operators which commute with $\mathcal{H}^{(s)}$ has only recently been found [16]. The simpler case of the infinite chain $N \to \infty$, $h(j) \to [\sinh(\pi/\kappa)/\sinh(\pi j/\kappa)]^2$ has been considered in detail in [17]. As for finite N, the description of the spectrum has been performed only for simplest two- and three-magnon excitations over ferromagnetic vacuum [10, 18, 19].

The aim of this letter is to demonstrate the remarkable correspondence between the highest-weight eigenstates of the lattice Hamiltonian with the elliptic exchange (5) and the double quasiperiodic meromorphic eigenfunctions of the Calogero–Moser operator (4) which allows one to formulate Bethe-ansatz-type equations for calculating the whole spectrum.

The Hamiltonian (1) commutes with the operator of total spin $S = \frac{1}{2}\sum_{j=1}^{N}\sigma_j$. Then the eigenproblem for it is decomposed into the problems in the subspaces formed by the common eigenvectors of S_3 and S^2 such that $S = S_3 = N/2 - M$, $0 \leqslant M \leqslant [N/2]$,

$$\mathcal{H}^{(s)}|\psi^{(M)}\rangle = E_M|\psi^{(M)}\rangle. \tag{6}$$

The eigenvectors $|\psi^{(M)}\rangle$ are written in the usual form

$$|\psi^{(M)}\rangle = \sum_{n_1\cdots n_M}^{N}\psi_M(n_1\cdots n_M)\prod_{\beta=1}^{M}s_{n_\beta}^-|0\rangle \tag{7}$$

where $|0\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$ is the ferromagnetic ground state with all spins up and the summation is taken over all combinations of integers $\{n\} \leqslant N$ such that $\prod_{\mu<\nu}^{M}(n_\mu - n_\nu) \neq 0$. Substitution of (7) in (6) results in the lattice Schrödinger equation for the completely symmetric wavefunction ψ_M

$$\sum_{s\neq n_1,\ldots,n_M}^{N}\sum_{\beta=1}^{M}\wp_N(n_\beta - s)\psi_M(n_1,\ldots,n_{\beta-1},s,n_{\beta+1},\ldots,n_M)$$

$$+\left[\sum_{\beta\neq\gamma}^{M}\wp_N(n_\beta-n_\gamma)-\mathcal{E}_M\right]\psi_M(n_1,\ldots,n_M)=0. \tag{8}$$

The eigenvalues $\{E_M\}$ are given by

$$E_M=J\left(\frac{\omega}{\pi}\sin\frac{\pi}{\omega}\right)^2\left\{\mathcal{E}_M+\frac{2}{\omega}\left[\frac{2M(2M-1)-N}{4}\zeta_N\left(\frac{\omega}{2}\right)-M\zeta_1\left(\frac{\omega}{2}\right)\right]\right\} \tag{9}$$

where $\zeta_1(x)$ is the Weierstrass zeta function defined on the torus $T_1=\mathbf{C}/\mathbf{Z}+\mathbf{Z}\omega$.

To find the solutions to (8), let us consider the following ansatz for ψ_M:

$$\psi_M(n_1,\ldots,n_M)=\sum_{P\in\pi_M}\varphi_M^{(p)}(n_{P1},\ldots,n_{PM}) \tag{10}$$

$$\varphi_M^{(p)}(n_1,\ldots,n_M)=\exp\left(-\mathrm{i}\sum_{\nu=1}^{M}p_\nu n_\nu\right)\chi_M^{(p)}(n_1,\ldots,n_M) \tag{11}$$

where π_M is the group of all permutations $\{P\}$ of the numbers from 1 to N and $\chi_M^{(p)}$ is the solution to the *continuum* quantum many-particle problem

$$\left[-\frac{1}{2}\sum_{\beta=1}^{M}\frac{\partial^2}{\partial x_\beta^2}+\sum_{\beta\neq\lambda}^{M}\wp_N(x_\beta-x_\lambda)-\mathrm{E}_M(p)\right]\chi_M^{(p)}(x_1,\ldots,x_M)=0. \tag{12}$$

It is specified up to a normalization factor by the particle pseudomomenta $(p_1,\ldots,p_M)$. The standard argumentation of the Floquet–Bloch theory shows that due to perodicity of the potential term in (12) $\chi_M^{(p)}$ obeys the quasiperiodicity conditions [18]

$$\chi_M^{(p)}(x_1,\ldots,x_\beta+N,\ldots,x_M)=\exp(\mathrm{i}p_\beta N)\chi_M^{(p)}(x_1,\ldots,x_M) \tag{13}$$

$$\chi_M^{(p)}(x_1,\ldots,x_\beta+\omega,\ldots,x_M)=\exp(q_\beta(p)+\mathrm{i}p_\beta\omega)\chi_M^{(p)}(x_1,\ldots,x_M)$$
$$0\leqslant \operatorname{Im}m(q_\beta)<2\pi\quad 1\leqslant\beta\leqslant M. \tag{14}$$

The eigenvalue $\mathrm{E}_M(p)$ is some symmetric function of $(p_1,\ldots,p_M)$. The set $\{q_\beta(p)\}$ is also completely determined by $\{p\}$. In this letter I do not refer to the explicit form of these functions, which is still unknown for $M>3$.

The structure of the singularity of $\wp_N(x)$ at $x=0$ implies that $\chi_M^{(p)}$ can be presented in the form

$$\chi_M^{(p)}=\frac{F^{(p)}(x_1,\ldots,x_M)}{G(x_1,\ldots,x_M)}\qquad G(x_1,\ldots,x_M)=\prod_{\alpha<\beta}^{M}\sigma_N(x_\alpha-x_\beta) \tag{15}$$

where $\sigma_N(x)$ is the Weierstrass sigma function on the torus T_N. The only simple zero of $\sigma_N(x)$ on T_N is located at $x=0$. Thus $[G(x_1,\ldots,x_M)]^{-1}$ absorbs all the singularities of $\chi_M^{(p)}$ on the hypersurfaces $x_\alpha=x_\beta$. The numerator $F^{(p)}$ in (15) is analytic on $(T_N)^M$ and obeys the equation

$$\sum_{\alpha=1}^{M}\frac{\partial^2F^{(p)}}{\partial x_\alpha^2}+\left[2E_M(p)-\frac{M}{2}\sum_{\alpha\neq\beta}^{M}(\wp_N(x_\alpha-x_\beta)-\zeta_N^2(x_\alpha-x_\beta))\right]F^{(p)}$$
$$=\sum_{\alpha\neq\beta}\zeta_N(x_\alpha-x_\beta)\left(\frac{\partial F^{(p)}}{\partial x_\alpha}-\frac{\partial F^{(p)}}{\partial x_\beta}\right). \tag{16}$$

The regularity of the left-hand side of (16) as $x_\mu\to x_\nu$ implies that

$$\left(\frac{\partial}{\partial x_\mu}-\frac{\partial}{\partial x_\nu}\right)F^p(x_1,\ldots,x_M)|_{x_\mu=x_\nu}=0 \tag{17}$$

for any pair (μ, ν).

The remarkable fact is that the properties (13–15), (17) of $\chi_M^{(p)}$ allow one to validate the ansatz (10), (11) for the eigenfunctions of the lattice Schrödinger equation (8). Substitution of (10) in (8) yields

$$\sum_{P\in\pi_M}\left\{\sum_{\beta=1}^{M}\mathcal{S}_\beta(n_{P1},\ldots,n_{PM})+\left[\sum_{\beta\neq\gamma}^{M}\wp_N(n_{P\beta}-n_{P\gamma})-\mathcal{E}_M\right]\varphi_M^{(p)}(n_{P1},\ldots,n_{PM})\right\}=0 \tag{18}$$

where

$$\mathcal{S}_\beta(n_{P1},\ldots,n_{PM})=\sum_{s\neq n_{P1},\ldots,n_{PM}}^{N}\wp_N(n_{P\beta}-s)\hat{Q}_\beta^{(s)}\varphi_M^{(p)}(n_{P1},\ldots,n_{PM}). \tag{19}$$

The operator $\hat{Q}_\beta^{(s)}$ in (19) replaces the βth argument of the function of M variables to s.

To calculate the sum (19), let us introduce, following the consideration of the hyperbolic exchange in [17], the function of one complex variable x:

$$W_P^{(\beta)}(x)=\sum_{s=1}^{M}\wp_N(n_{P\beta}-s-x)\hat{Q}_\beta^{(s+x)}\varphi_M^{(p)}(n_{P1},\ldots,n_{PM}). \tag{20}$$

As a consequence of (11), (13), (14), it obeys the relations

$$W_P^{(\beta)}(x+1)=W_P^{(\beta)}(x)\qquad W_P^{(\beta)}(x+\omega)=\exp(q_\beta(p))W_P^{(\beta)}(x). \tag{21}$$

The only singularity of $W_P^{(\beta)}$ on the torus $T_1=\mathbf{C}/\mathbf{Z}+\mathbf{Z}\omega$ is located at the point $x=0$. It arises from the terms in (20) with $s=n_{P1},\ldots,n_{PM}$. The Laurent decomposition of (20) near $x=0$ has the form

$$W_P^{(\beta)}(x)=w_{-2}x^{-2}+w_{-1}x^{-1}+w_0+\mathrm{O}(x). \tag{22}$$

The explicit expressions for w_{-i} can be found from (20):

$$w_{-2}=\varphi_M^{(p)}(n_{P1},\ldots,n_{PM}) \tag{23a}$$

$$w_{-1}=\frac{\partial}{\partial n_{P\beta}}\varphi_M^{(p)}(n_{P1},\ldots,n_{PM})+(-1)^P G(n_1,\ldots,n_M)\sum_{\lambda\neq\beta}T_{\beta\lambda}(n_{P1},\ldots,n_{PM})\hat{Q}_\beta^{(n_{P\lambda})}$$
$$\times\exp\left(-\mathrm{i}\sum_{\nu=1}^{M}p_\nu n_{P\nu}\right)F^{(p)}(n_{P1},\ldots,n_{PM}) \tag{23b}$$

$$w_0=\mathcal{S}_\beta(n_{P1},\ldots,n_{PM})+\frac{1}{2}\frac{\partial^2}{\partial n_{P\beta}}\varphi_M^{(p)}(n_{P1},\ldots,n_{PM})+(-1)^P G(n_1,\ldots,n_M)$$
$$\times\sum_{\lambda\neq\beta}T_{\beta\lambda}(n_{P1},\ldots,n_{PM})$$
$$\left[U_{\beta\lambda}(n_{P1},\ldots,n_{PM})\hat{Q}_\beta^{(n_{P\lambda})}+\wp_N(n_{P\beta}-n_{P\lambda})\partial\hat{Q}_\beta^{(n_{P\lambda})}\right]$$
$$\times\exp\left(-\mathrm{i}\sum_{\nu=1}^{M}p_\nu n_{P\nu}\right)F^{(p)}(n_{P1},\ldots,n_{PM}) \tag{23c}$$

where

$$T_{\beta\lambda}(n_{P1},\ldots,n_{PM}) = \sigma_N(n_{P\lambda}-n_{P\beta})\prod_{\rho\neq\beta,\lambda}^{M}\frac{\sigma_N(n_{P\rho}-n_{P\beta})}{\sigma_N(n_{P\rho}-n_{P\lambda})}$$

$$U_{\beta\lambda}(n_{P1},\ldots,n_{PM}) = \wp_N'(n_{P\lambda}-n_{P\beta}) - \wp_N(n_{P\beta}-n_{P\lambda})\sum_{\rho\neq\beta,\lambda}\zeta_N(n_{P\rho}-n_{P\lambda}).$$

Here $(-1)^P$ means that the parity of the permutation P and the action of the operator $\partial\hat{Q}_\beta^{(n_{P\lambda})}$ on the function Y of M variables is defined as

$$\partial Q_\beta^{(n_{P\lambda})}Y(z_1,\ldots,z_M) = \frac{\partial}{\partial z_\beta}Y(z_1,\ldots,z_M)|_{z_\beta=n_{P\lambda}}. \tag{24}$$

The next step consists of writing the explicit expression for the function $W_P^{(\beta)}(x)$ obeying the relations (21) and (22) [17]:

$$\begin{aligned} W_P^{(\beta)}(x) = \exp(a_\beta x)\frac{\sigma_1(r_\beta+x)}{\sigma_1(r_\beta-x)}\{&w_{-2}(\wp_1(x)-\wp_1(r_\beta)+(w_{-2}(a_\beta+2\zeta_1(r_\beta))-w_{-1})\\ &\times[\zeta_1(x-r_\beta)-\zeta_1(x)+\zeta_1(r_\beta)-\zeta_1(2r_\beta)]\}. \end{aligned} \tag{25}$$

The Weierstrass functions $\wp_1$, ζ_1 and σ_1 in (25) are defined on the torus T_1 and the parameters a_β, r_β are chosen so as to satisfy the conditions (21):

$$a_\beta = (\pi\mathrm{i})^{-1}q_\beta(p)\zeta_1(\tfrac{1}{2}) \qquad r_\beta = -(4\pi\mathrm{i})^{-1}q_\beta(p).$$

By expanding equation (25) in powers of x one can find w_0 in terms of w_{-2}, w_{-1}, q_β and obtain the explicit expression for $S_\beta(n_{P1},\ldots,n_{PM})$ with the use of (23*a*)–(23*c*). It turns out that equation (18) can be recast in the form

$$\begin{aligned} \sum_{P\in\pi_M}\Bigg[-\frac{1}{2}\sum_{\beta=1}^{M}\left(\frac{\partial}{\partial n_{P\beta}}-f_\beta(p)\right)^2 &+ \sum_{\beta\neq\gamma}^{M}\wp_N(n_{P\beta}-n_{P\gamma}) - \mathcal{E}_M\\ &+\sum_{\beta=1}^{M}\varepsilon_\beta(p)\Bigg]\varphi^{(p)}(n_{P1},\ldots,n_{PM})\\ &= \tfrac{1}{2}G(n_1,\ldots,n_M)\sum_{P\in\pi_M}(-1)^P\sum_{\beta\neq\lambda}\big[Z_{\beta\lambda}(n_{P1},\ldots,n_{PM})\\ &+ Z_{\lambda\beta}(n_{P1},\ldots,n_{PM})\big] \end{aligned} \tag{26}$$

where

$$f_\beta(p) = (\pi\mathrm{i})^{-1}q_\beta(p)\zeta_1(\tfrac{1}{2}) - \zeta_1((2\pi\mathrm{i})^{-1}q_\beta(p)) \tag{27}$$

$$\varepsilon_\beta(p) = \tfrac{1}{2}\wp_1((2\pi\mathrm{i})^{-1}q_\beta(p)) \tag{28}$$

and $Z_{\beta\lambda}(n_{P1},\ldots,n_{PM})$ is defined by the relation

$$\begin{aligned} Z_{\beta\lambda}(n_{P1},\ldots,n_{PM}) = T_{\beta\lambda}(n_{P1},\ldots,n_{PM})\Big[&U_{\beta\lambda}(n_{P1},\ldots,n_{PM})\hat{Q}_\beta^{(n_{P\lambda})} + \wp_N(n_{P\lambda}-n_{P\beta})\\ &\times(\partial\hat{Q}_\beta^{(n_{P\lambda})} - f_\beta(p)\hat{Q}_\beta^{(n_{P\lambda})})\Big]\exp\left(-\mathrm{i}\sum_{\nu=1}^{M}p_\nu n_{P\nu}\right)F^{(p)}(n_{P1},\ldots,n_{PM}). \end{aligned} \tag{29}$$

Turning to the definition (11) of $\varphi^{(p)}$, one observes that each term on the left-hand side of (26) has the same structure as the left-hand side of the many-particle Schrödinger equation (12), and vanishes if $\mathcal{E}_M$ and $f_\beta(p)$ are chosen to be

$$f_\beta(p) = -\mathrm{i}p_\beta \qquad \beta = 1, \ldots, M \tag{30}$$

$$\mathcal{E}_M = \mathrm{E}_M(p) + \sum_{\nu=1}^{M} \varepsilon_\beta(p). \tag{31}$$

Now let us prove that that the right-hand side of (26) also vanishes. The crucial observation is that the sum over permutations in it can be recast in the form

$$\sum_{P\in\pi_M} (-1)^P \sum_{\beta\neq\lambda} [Z_{\beta\lambda}(n_{P1}, \ldots, n_{PM}) - Z_{\lambda\beta}(n_{PR1}, \ldots, n_{PRM})]$$

where R is the transposition $(\beta \leftrightarrow \lambda)$ which leaves the other numbers from 1 to M unchanged. The term in square brackets is simplified drastically with the use of the identities

$$T_{\lambda\beta}(n_{PR1}, \ldots, n_{PRM}) = T_{\beta\lambda}(n_{P1}, \ldots, n_{PM})$$

$$U_{\lambda\beta}(n_{PR1}, \ldots, n_{PRM}) = U_{\beta\lambda}(n_{P1}, \ldots, n_{PM})$$

$$\hat{Q}_\lambda^{(n_{P\beta})} F(n_{PR1}, \ldots, n_{PRM}) = \hat{Q}_\beta^{(n_{P\lambda})} F(n_{P1}, \ldots, n_{PM}).$$

Taking into account relations (29), (30), one finds

$$\begin{aligned} Z_{\beta\lambda}&(n_{P1}, \ldots, n_{PM}) - Z_{\lambda\beta}(n_{PR1}, \ldots, n_{PRM}) \\ &= T_{\beta\lambda}(n_{P1}, \ldots, n_{PM}) \wp_N(n_{P\lambda} - n_{P\beta}) \\ &\quad \times \exp\left[-\mathrm{i}\left((p_\beta + p_\lambda)n_{P\lambda} + \sum_{\rho\neq\beta,\lambda}^{M} p_\rho n_{P\rho}\right)\right]\left(\frac{\partial}{\partial n_{P\beta}} - \frac{\partial}{\partial n_{P\lambda}}\right) \\ &\quad \times F^{(p)}(n_{P1}, \ldots, n_{PM})|_{n_{P\beta}=n_{P\lambda}}. \end{aligned} \tag{32}$$

The last factor in (32) vanishes due to condition (17) imposed by the regularity of the left-hand side of the Schrödinger equation (16).

It remains for us to show that the states of the spin lattice given by (7) with the functions ψ_M of the form (10), (11) are highest-weight states with $S = S_3$. This statement is equivalent to the relation $\mathbf{S}_+|\psi^{(M)}\rangle = 0$, which can be rewritten as

$$\sum_{\beta=1}^{M} \sum_{P\in\pi_M^{(\beta)}} \sum_{s\neq n_1,\ldots,n_{M-1}} \hat{Q}_\beta^{(s)} \varphi_M^{(p)}(n_{P1}, \ldots, n_{PM}) = 0 \tag{33}$$

where $\{\pi_M^{(\beta)}\}$ are the subsets of π_M: $P \in \pi_M^{(\beta)} \leftrightarrow P\beta = M$. The sums over s in (33) can be reduced and presented in closed form by using the technique described above. It turns out that the left-hand side of (33) contains factors similar to the last factor in (32) and vanishes due to condition (17).

The descendant states with $S_3 < S$ are obtained by acting with $\mathbf{S}_-$ on the basic states $|\psi^{(M)}\rangle$ (7). Thus the present consideration allows one, in principle, to reproduce all the eigenvectors of $\mathcal{H}^{(s)}$ for the exchange (5), as has been done by Bethe [1] for nearest-neighbour spin coupling. Equations (30) for the pseudomomenta $\{p\}$ constitute the analogue of the usual Bethe ansatz. The spectrum is given by relations (9) and (31).

In conclusion, it has been demonstrated that the procedure of exact diagonalization of the lattice Hamiltonian with non-nearest-neighbour elliptic exchange can be reduced in each sector of the Hilbert space with given magnetization to the construction of the special double quasiperiodic eigenfunctions of the many-particle Calogero–Moser problem on a continuous line. The Bethe-ansatz-type equations appear very naturally as a set of restrictions on the particle pseudomomenta. The proof of this correspodence between lattice and continuum integrable models is based only on analytic properties of the eigenfunctions. One can expect that the set of spin lattice states constructed in this way is complete. This is supported by an exact analytic proof in the two-magnon case.

The analysis of the explicit form of equations (30) available for $M = 2, 3$ shows that the spectrum of the lattice Hamiltonian with the exchange (5) is *not* additive, rather it is given in terms of pseudomomenta $\{p\}$ or phases which parametrize the sets $\{p, q\}$ [10, 19]. The problem of finding appropriate set of parameters which gives the 'separation' of the spectrum remains open. It would be also of interest to consider various limits ($N \to \infty, \kappa \to 0, \infty$) so as to recover the results of [1, 3, 17] and prove the validity of the approximate methods of the asymptotic Bethe ansatz after finding the explicit form of the functions $q_\beta(p)$ and $E_M(p)$.

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References

[1] Bethe H 1931 *Z. Phys.* **71** 205
[2] Haldane F D M 1988 *Phys. Rev. Lett.* **60** 635
Shastry B S 1988 *Phys. Rev. Lett.* **60** 639
[3] Haldane F D M 1991 *Phys. Rev. Lett.* **66** 1529
[4] Faddeev L D 1984 *Integrable Models of 1+1 Dimensional Quantum Field Theory* (Amsterdam: Elsevier)
[5] Bernard D, Gaudin M, Haldane F D M and Pasquier V 1993 *J. Phys. A: Math. Gen.* **26** 5219
[6] Takahashi M 1971 *Prog. Theor. Phys.* **46** 401
[7] Lieb E H and Liniger M 1963 *Phys. Rev.* **130** 1605
[8] Takahashi M 1994 *Prog. Theor. Phys.* **91** 1
[9] Kawakami N 1994 *Prog. Theor. Phys.* **91** 189
[10] Inozemtsev V I 1990 *J. Stat. Phys.* **59** 1143
[11] Calogero F 1975 *Lett. Nuovo Cimento* **13** 411
[12] Moser J 1975 *Adv. Math.* **16** 1
[13] Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 313
[14] Etingof P I 1993 Quantum integrable systems and representations of Lie algebras *Preprint* hep-th 9311132
[15] Etingof P I and Kirillov A A Jr 1994 *Duke Math. J.* **74** 585
[16] Inozemtsev V I 1995 Invariants of linear combinations of transpositions *ISSP Preprint* 2928 (*Lett. Math. Phys.* to appear)
[17] Inozemtsev V I 1992 *Commun. Math. Phys.* **148** 359
[18] Dittrich J and Inozemtsev V I 1993 *J. Phys. A: Math. Gen.* **26** L753
[19] Inozemtsev V I 1995 Solution to three-magnon problem for 1D quantum chains with elliptic exchange *ISSP Preprint* 2947